JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 3 MAY-JUNE, 1962

Transient Response of a Dipole Antenna

SAMUEL P. MORGAN
Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey
(Received January 30, 1962)

The transient current in a long dipole antenna excited by a step-function voltage across an infinitesimal center gap has recently been calculated by Wu. A simpler derivation of Wu's result is given using double Fourier transformation.

In a recent paper,[1] Wu calculates the current in a long dipole antenna driven by a voltage impressed across an infinitesimal center gap. The impressed voltage is a step function of both space and time, and the current is found up until the time that the discontinuity spreading out from the gap first reaches the ends of the dipole. Since Wu's analysis, while ingenious, is somewhat roundabout, the following more direct solution may be of interest.

In a cylindrical coordinate system (ρ, φ, z), the resultant electric field at the surface of the antenna, which is the negative of the applied electric field, may be expressed as[2]

$$\begin{aligned} \boldsymbol{E}_z(a, z, t) &= -\delta(z)U(t) \\ &= \lim_{s\to 0} \frac{1}{4\pi^2} \int_{C_\gamma} \frac{\sinh \frac{1}{2}\gamma s}{\frac{1}{2}\gamma s} e^{\gamma z}\, d\gamma \int_{C_p} \frac{e^{pct}}{p}\, dp, \qquad (1) \end{aligned}$$

where c is the velocity of light, and the contours C_γ and C_p are up the imaginary axes, with C_p indented to the right of $p = 0$.

The electromagnetic field will be transverse magnetic and derivable from a stream function $\Pi(\rho, z, t)$ which satisfies the scalar wave equation.[2] Such a function may be written in the form

$$\begin{aligned} &\Pi(\rho, z, t) \\ &= \frac{1}{4\pi^2} \int_{C_\gamma} \int_{C_p'} A(p, \gamma) \frac{K_0(\lambda\rho)}{K_0(\lambda a)} e^{\gamma z + pct}\, dp\, d\gamma, \qquad (2) \end{aligned}$$

where

$$\lambda = (p^2 - \gamma^2)^{\frac{1}{2}}. \qquad (3)$$

In order to represent outgoing or damped waves at infinity, λ must lie in the fourth quadrant or on its boundaries when $\operatorname{Im} p < 0$, and in the first quadrant or on its boundaries when $\operatorname{Im} p > 0$. To carry out the p integration when $\gamma = i\beta$, where $\beta \neq 0$, take the contour C_p' shown in Fig. 1, indented to pass to the right of $p = \pm i\beta$. It is easy to verify that if λ is taken to be negative imaginary when p lies on C_p' below $-i\,|\beta|$, then λ describes the contour C_λ' shown in Fig. 1 as p describes C_p', and so only outgoing or damped waves are included in the integral (2). If $\beta = 0$, then C_p' coincides with C_p and $\lambda = p$. To make the Bessel functions single valued, the λ plane may be cut along the negative real axis.

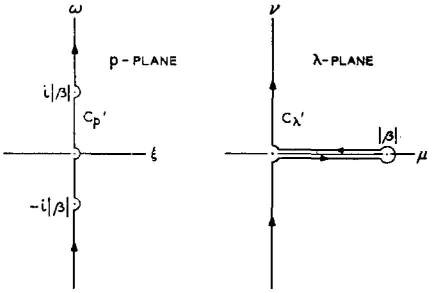

FIG. 1. Contours of integration.

The electric field corresponding to (2) at the surface of the antenna is

$$\begin{aligned} &E_z(a, z, t) \\ &= -\frac{1}{4\pi^2} \int_{C_\gamma} \int_{C_p'} A(p, \gamma) \frac{\eta_0 \lambda^2}{p} e^{\gamma z + pct}\, dp\, d\gamma, \qquad (4) \end{aligned}$$

where η_0 is the characteristic impedance of free space, so that the function $A(p, \gamma)$ may be determined by comparison with (1). The antenna current, which is proportional to the surface magnetic field, is given by

$$\begin{aligned} I(z, t) &= -2\pi a \left.\frac{\partial \Pi}{\partial \rho}\right|_{\rho = a} \\ &= -\frac{a}{2\pi\eta_0} \int_{C_\gamma} \int_{C_p'} \frac{K_1(\lambda a)}{\lambda K_0(\lambda a)} e^{\gamma z + pct}\, dp\, d\gamma, \qquad (5) \end{aligned}$$

where the convergence factor $(\sinh \frac{1}{2}\gamma s)/\frac{1}{2}\gamma s$ has been dropped on the assumption that z and t are not simultaneously zero.

To simplify the expression for $I(z, t)$, we observe that the integrand of (5) does not have a pole at $p = 0$, so the center part of C_p' may be deformed into an infinite semicircle in the left half-plane, over which the integrand vanishes when $t > 0$. The final stage of deformation is shown as C_p'' in Fig. 2, where nothing is left but the integrals around the branch cuts from $p = \pm i\,|\beta|$ to $\pm i\infty$. The corresponding contour in the λ plane is C_λ''. It is easy to show that during the deformation C_λ does not leave the right half-plane, in which the function $K_0(\lambda a)$ has no zeros; furthermore, the ratio $K_1(\lambda a)/K_0(\lambda a)$ approaches 1 as $|\lambda|$ approaches ∞. The contributions from the circles around $p = \pm i\,|\beta|$ can be shown to be of the order of $1/|\log \delta|$, where δ is the radius of these circles in the p plane, so the contributions vanish as $\delta \rightarrow 0$.

---

[1] T. T. Wu, J. Math. Phys. **2**, 892 (1961).

[2] S. A. Schelkunoff, *Electromagnetic Waves* (D. van Nostrand Company, Inc., New York, 1943), pp. 34–35 and 375–377.

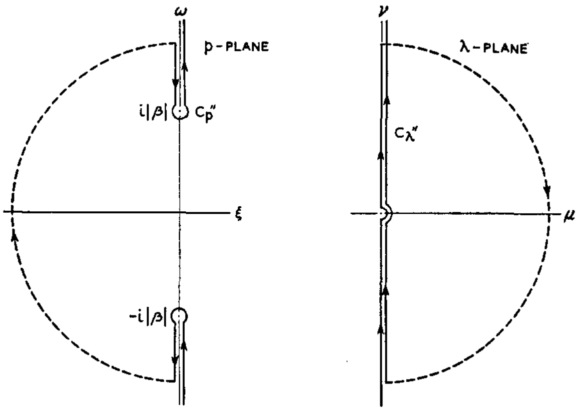

Fig. 2. Contours of integration.

If we now set

$$\gamma = i\beta, \qquad p = i\omega, \qquad \nu = (\omega^2 - \beta^2)^{\frac{1}{2}}, \tag{6}$$

where ν is real, we can write down the four integrals along the branch lines. Expressing the modified Bessel functions $K_{0,1}(\pm i\nu a)$ in terms of Hankel functions and combining terms gives

$$\begin{aligned} I(z, t) &= \frac{ai}{\pi\eta_0}\int_{-\infty}^{\infty}\int_{|\beta|}^{\infty}\left[\frac{H_1^{(1)}(\nu a)}{\nu H_0^{(1)}(\nu a)} - \frac{H_1^{(2)}(\nu a)}{\nu H_0^{(2)}(\nu a)}\right]\sin(\omega ct)e^{i\beta z}\,d\omega\,d\beta \\ &= \frac{4}{\pi^2\eta_0}\int_{-\infty}^{\infty}\int_{|\beta|}^{\infty}\frac{\sin(\omega ct)e^{i\beta z}\,d\omega\,d\beta}{\nu^2[J_0^2(\nu a) + N_0^2(\nu a)]}, \end{aligned} \tag{7}$$

where in the last step we have used the Wronskian relationship between the Hankel functions. Changing the inner variable of integration from ω to ν and then inverting the order of integration and using the result[3]

$$\int_{-\infty}^{\infty}\frac{\sin[ct(\beta^2 + \nu^2)^{\frac{1}{2}}]}{(\beta^2 + \nu^2)^{\frac{1}{2}}}e^{i\beta z}\,d\beta = \begin{cases} 0 & \text{if} \quad |z| > ct, \\ \pi J_0[\nu(c^2t^2 - z^2)^{\frac{1}{2}}] & \text{if} \quad |z| < ct, \end{cases} \tag{8}$$

we find that the antenna current vanishes for $|z| > ct$. For $|z| < ct$ it is given by

$$I(z, t) = \frac{4}{\pi\eta_0}\int_0^{\infty}\frac{J_0[\nu(c^2t^2 - z^2)^{\frac{1}{2}}]}{J_0^2(\nu a) + N_0^2(\nu a)}\frac{d\nu}{\nu}, \tag{9}$$

a result which is clearly equivalent to Wu's Eq. (17).

---

[3] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), p. 118.